- ## Standard 1: Make sense of problems and persevere in solving them

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Classroom Observations:

Teachers who are developing students' capacity to "make sense of problems and persevere in solving them" develop ways of framing mathematical challenges that are clear and explicit, and then check in repeatedly with students to help them clarify their thinking and their process. An early childhood teacher might ask her students to work in pairs to evaluate their approach to a problem, telling a partner to describe their process, saying "what [they] did, and what [they] might do next time." A middle childhood teacher might post a set of different approaches to a solution, asking students to identify "what this mathematician was thinking or trying out" and evaluating the success of the strategy. An early adolescence teacher might have students articulate a specific way of laying out the terrain of a problem and evaluating different starting points for solving. A teacher of adolescents and young adults might frame the task as a real-world design conundrum, inviting students to engage in a "tinkering" process of working toward mathematical proof, changing course as necessary as they develop their thinking. Visit the video excerpts below to view multiple examples of teachers engaging students in sense-making and mathematical perseverance

- ## Standard 2: Reason abstractly and quantitatively

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to de-contextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Classroom Observations:

Teachers who are developing students' capacity to "reason abstractly and quantitatively" help their learners understand the relationships between problem scenarios and mathematical representation, as well as how the symbols represent strategies for solution. A middle childhood teacher might ask her students to reflect on what each number in a fraction represents as parts of a whole. A different middle childhood teacher might ask his students to discuss different sample operational strategies for a patterning problem, evaluating which is the most efficient and accurate means of finding a solution. Visit the video excerpts below to view these teachers engaging their students in abstract and quantitative reasoning.

- ## Standard 3: Construct viable arguments and critique the reasoning of others

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Classroom Observations

Teachers who are developing students' capacity to "construct viable arguments and critique the reasoning of others" require their students to engage in active mathematical discourse. This might involve having students explain and discuss their thinking processes aloud, or signaling agreement/disagreement with a hand signal. A middle childhood teacher might post multiple approaches to a problem and ask students to identify plausible rationales for each approach as well as any mistakes made by the mathematician. An early adolescence teacher might post a chart showing a cost-analysis comparison of multiple DVD rental plans and ask his students to formulate and defend a way of showing when each plan

becomes most economical. A teacher of adolescents and young adults might actively engage her students in extended conjecture about conditions for proof in the construction of quadrilaterals, testing their assumptions and questioning their approaches. Visit the video excerpts below to view multiple examples of teachers engaging students in formulating, critiquing and defending arguments of mathematical reasoning.

- ## Standard 4: Model with mathematics

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Classroom Observations:

Teachers who are developing students' capacity to "model with mathematics" move explicitly between real-world scenarios and mathematical representations of those scenarios. A middle childhood teacher might pose a scenario of candy boxes containing multiple flavors to help students identify proportions and ratios of flavors and ingredients. An early adolescence teacher might represent a comparison of different DVD rental plans using a table, asking the students whether or not the table helps directly compare the plans or whether elements of the comparison are omitted. A teacher of adolescents and young adults might pose a "kite factory" scenario, in which advanced students are asked to determine the conditions for always creating a particular shape of kite given the dimensions of the diagonals and the angle of intersection. Visit the video excerpts below to view multiple examples of teachers engaging students in mathematical modeling.

- ## Standard 5: Use appropriate tools strategically

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Classroom Observations:

Teachers who are developing students' capacity to "use appropriate tools strategically" make clear to students why the use of manipulatives, rulers, compasses, protractors, and other tools will aid their problem solving processes. A middle childhood teacher might have his students select different color tiles to show repetition in a patterning task. A teacher of adolescents and young adults might have established norms for accessing tools during the students' group "tinkering processes," allowing students to use paper strips, brass fasteners, and protractors to create and test quadrilateral "kite" models. Visit the video excerpts below to view multiple examples of these teachers

- ## Standard 6: Attend to precision

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

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• Standard 7: Look for and make use of structure

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression x2 + 9x + 14, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see 5 – 3(x – y)2 as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Classroom Observations:

Teachers who are developing students' capacity to "look for and make use of structure" help learners identify and evaluate efficient strategies for solution. An early childhood teacher might help students identify why using "counting on" is preferable to counting each addend by one, or why multiplication or division can be preferable to repeated addition or subtraction. A middle childhood teacher might help his students discern patterns in a function table to "guess my rule." A teacher of adolescents and young adults might focus on exploring geometric processes through patterns and proof. Visit the video excerpts below to view multiple examples of teachers engaging students in identifying and making use of mathematical structure.

• Standard 8: Look for and express regularity in repeated reasoning

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation (y - 2)/(x - 1) = 3. Noticing the regularity in the way terms cancel when expanding (x-1)(x+1),(x-1)(x2+x+1),and(x-1)(x3 +x2+x+1)might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Classroom Observations:

These mathematical practice standards do not provide field based evidence of Standard 8. This standard is particularly difficult to capture in video from one lesson. Engaging in repeated reasoning may emerge over many weeks, so that students learning to divide fractions, students may gradually attain insight into the underlying concepts over many days or weeks. In the end, students arrive at the algorithm or formula through discovery, but in a longitudinal discovery process, and so sharing the "eureka" moment from a single lesson will not reveal the process that led to the insight. Because of the nature of Inside Mathematics' engagement in classrooms, we did not embed ourselves in rooms for many weeks as might be required to surface evidence of standard 8 in the field.